Stability Analysis of a Multi-class Retrial Queue with General Retrials and Classical Retrial Policy

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***Abstract*—We deal with a single server multi-class retrial model, feed by Poisson input. The system is considered under classical retrial policy, while inter-retrial times are class dependent and generally distributed. Such systems have various applications like multi-access protocols or cellular mobile networks, where blocked messages are sent again after some waiting period.**

We rely on regenerative approach and results from renewal theory to obtain the stability criterion of the system under consideration and present some simulation results, to illustrate that obtained condition could be extended to the case with general input.

I. Introduction

In this paper we consider a multi-class retrial system with the only server. The total input stream is assumed to be Poisson, while marginal input rates and service times are class-dependent. The model belongs to the classical retrial policy. Thus, blocked customer stays in the corresponding class orbit for a *generally distributed retrial time* independently of other customers, and summary retrial rate is proportional to the number of customers in all orbits.

Retrial queues were naturally arisen from various applications with multiple access like telephone networks [1] or call centers [2] and now are successfully used in modeling of a large number of a modern objects like wireless telecommunication systems, cellular mobile networks or multi-access protocols, where blocked packets are sent again after some waiting period.

Retrial queuing models are widely studied in the literature. Regarding to the overview of retrial theory it is worth mention the basic books [3], [4] and, for instance, quite recent survey papers [5], [6].

Obviously, that most of research in this field is related to more widespread single-class Marcovian models with exponential retrials. In that case authors may establish explicit statements for stability conditions or stationary characteristics (see, for instance [7], [8]). The paper [9], which contains stability results for a single-class model with the only server, was one of the first works, where authors considered non-exponential distribution of inter-retrial times. Later, in [10] the analysis was extended to a multiserver single-class system, the research was based on the regenerative approach. The similar model was considered in [11], where the analysis relayed on the fluid limit approach.

In this paper we construct a single server multi-class model with general retrials. Our goal is to obtain stability condition under the assumption, that input stream is Poisson. The research is based on the regenerative method, which is a strong instrument in stochastic analysis. The basic steps of the proof are relayed on preliminary results from papers [16] and [17], where authors considered multi-class retrial models with exponential and New Better than Used retrials, respectively. One of the key moments of the research in present paper is based on Lorden's inequality for embedded renewal process, associated with orbit stream. We obtain that in case the load coefficient is less than 1, the system is stable. The assumption that obtained condition guarantees the stability in case of general input (at least, bounded inter-arrival times) is confirmed by simulations.

The structure of the paper is the following. Section II contains a detailed description of the model under consideration and introduces it's regenerative structure. Then in Section III some previous stability results, related to particular cases of multi-class retrial systems, are presented. Next in Section IV the stability condition for a model with general retrials and unbounded inter-arrival times was established. The analysis was based on regenerative approach and results from renewal theory, namely we relied on Lorden's inequality for residual retrial time. Section V containes simulation results for the system with bounded input stream. The experiments illustrate that stability conditions, obtained in previous section, hold for the model with a general renewal input. Section VI concludes the paper.

II. Model description

In this section we construct a multi-class and single-server retrial model as follows. Consider a Poisson input with a total rate λ. Note, that arrivals join the system at instants $\{t_n,\, n \geq 1\}$, where inter-arrival times $\tau_n := t_n - t_{n-1}$ are independent and exponentially distributed with a mean $1/\lambda$. For simplicity, we assume $t_0 = 0$ and define the generic element of iid (independent identically distributed) sequence $\{\tau_n,\, n \geq 1\}$ by τ. Consider $K \geq 1$ different classes of customers with marginal input rates λ_i, $i = 1, \ldots, K$. Obviously

$$\lambda = \lambda_1 + \cdots + \lambda_K.$$

Moreover, we consider, $\lambda_i = p_i \lambda$, where $\mathbf{p} = (p_1, \ldots, p_K)$ is a given distribution.

Assume that service times in the system under consideration are class-dependent. Namely, the server, captured by class-i customer, would be realised in a random time, stochastically equivalent to $S^{(i)}$, while the distributions of $S^{(i)}$ for different

i may not coincide. If the n-th customer belongs to class-i_n, $i_n \in \{1, \dots, K\}$, we define it's service time by $S_n^{(i_n)}$. Note, that times $\{S_n^{(i_n)},\, n \ge 1\}$ are independent. Denote class-i load coefficient by $\rho_i = \lambda_i \mathsf{E} S^{(i)}$. Hence, the total system load coefficient is obtained as

$$\rho := \sum_{i=1}^{K} \rho_i.$$

We deal with a retrial system. Thus, if a new arrival finds the server busy, it does not lost, but joins to some kind of a virtual buffer, called *orbit*, and then, after a random time interval try to capture the server again. If the server is still busy, the orbital or *secondary* customer, immediately returns to the orbit, waits a random time and then makes new a attempts. (From this point of view, the arrivals from input stream are called *primary*.) As a customer is described by it's class characteristics, we consider, that the system has K different orbits. Class-i arrival joins the corresponding orbit, stays there for an interval distributed as $\xi^{(i)}$ and then try to attacks the server again. The i-th orbit rate is defined as follows

$$\gamma_i := 1/\mathsf{E}\xi^{(i)}, \quad i = 1, \dots, K.$$

The model under consideration obeys to *classical retrial policy*. Thus, all the secondary customers make independent attempts and the total (actual) orbit rate grows proportionally to sum orbit size. Note, that in case $\gamma_i = 0$ we obtain the loss system, while if $\gamma_i = \infty$, the orbit customer immediately captures the server, as in becomes empty, and the model is equivalent to the infinite-buffer queuing system. One more significant feature of a model under consideration is *non-exponential retrials*: class-i random orbit waiting times $\xi^{(i)}$, $i = 1, \dots, K$ are generally distributed, which makes the analysis more complicated.

Define the number of customers on orbit i just before time instant t by $N^{(i)}(t)$, thus for the summary orbit we have

$$N(t) := N^{(1)}(t) + \cdots + N^{(K)}(t), \quad t \ge 0.$$

Obviously, that the only reason of instability is the infinite growth of orbit size $N(t)$, as $t \to \infty$.

Our goal is to find the conditions, which guarantee the stable summary orbit. The basic stability analysis is relied on results from regeneration theory [12], [13], [14]. First we define by $\nu(t) \in \{0, 1\}$ the server state (number of customers on service) just before time instant t and assume zero initial state:

$$N(0) = 0, \quad \nu(0) = 0.$$

Then denote by $X(t) = \nu(t) + N(t), t \ge 0$ the process associated with the total number of customers in the system and it's discrete analogue at arrival instants by

$$X_n := X(t_n^-), \quad n \ge 1.$$

Next, we construct the sequence

$$\beta_n = \min_k \{k > \beta_{n-1} : X_k = 0\}, \quad n \ge 1,\, \beta_0 = 0,$$

which defines the numbers of customers, arrived into totally empty system. The instants $\{\beta_n\}$ are called *regeneration points* of the process $\{X_n\}$, while $\{X_n\}$ is a *regenerative process* with iid cycle lengths

$$\alpha_n = \beta_n - \beta_{n-1}, \quad n \ge 1.$$

The process $\{X_n\}$ is called *positive recurrent*, if $\mathsf{E}\alpha < \infty$, where α is a generic regeneration cycle length. The positive recurrence of the basic process X is equivalent to the stability of the model under consideration. By results from regeneration theory, for $\mathsf{E}\alpha < \infty$ its enough to show, that residual regeneration time

$$\beta(t_n) := \min_k \{t_{\beta_k} - t_n > 0\},\, n \ge 1 \tag{1}$$

does not convergence to infinity in distribution, as $n \to \infty$. This result takes place, if the orbit process $\{N_n, n \ge 1\}$ has negative drift. Note, that such a method was successfully applied for stability analysis of a multi-class and multi-server retrial system with exponential retrials in [16] and developed for the single-server system with Poisson input, where retrial waiting time $\xi^{(i)}$ belongs to the special subclass of *New Better than Used* (NBU) distributions (see [17]).

III. Preliminary results

In this section we present some stability results, obtained for less general retrial systems. In section IV we rely on such an analysis to establish stability conditions for the system under consideration.

The paper [16] deals with a multi-class $m \ge 1$ server retrial system with renewal input, general service time and exponential retrials. The authors obtained, that the condition

$$\rho < m$$

is indeed the stability criterion. The proof of necessity is based on the balance equation in terms of workload and does not rely on inter-retrial time distribution. Thus, in [17] the analysis was easily extended for the system with general retrials and in single-server case we have the following result.

Theorem 1. [17] *Assume that the K–class retrial system with generally distributed retrial times is stable (positive recurrent) and condition*

$$\max_{i=1,\dots,K} \mathsf{P}(\tau > S^{(i)}) > 0 \tag{2}$$

holds. Then

$$\rho < 1. \tag{3}$$

Note, that condition (2) automatically holds in case of Poisson input, as τ is unbound random variable.

To establish (3), in [17] we presented the total workload (the summary service times) $V(t)$, arrived to the system in $[0, t]$, as a sum of two components: $D(t)$ and $R(t)$ – departed and residual (on service and in the orbits) workload, respectively. As the system is stable, the summary orbit $N(t)$ is stochastically bounded. Thus, $R(t) = o(t)$, as $t \to \infty$. The departed in $[0, t]$ is presented via summary server idle time $I(t)$ as

$$D(t) = t - I(t).$$

By (2) and from regeneration theory, with probability 1:

$$\lim_{t\to\infty} \frac{I(t)}{t} > 0.$$

Moreover, denoting by $A_i(t)$ the number of class-i customers, arrived in $[0,\, t]$, and by Strong Law of Large Numbers

$$\frac{V(t)}{t} = \frac{1}{t}\sum_{i=1}^{K}\sum_{j=1}^{A_i(t)} S_j^{(i_j)} = \sum_{i=1}^{K}\frac{\sum_{j=1}^{A_i(t)} S_j^{(i_j)}}{A_i(t)}\frac{A_i(t)}{t}$$
$$\to \sum_{i=1}^{K}\mathsf{E}S^{(i)}\lambda_i = \rho, \quad t\to\infty.$$

Thus, the equation

$$V(t) = D(t) + R(t),$$

devided by $t \to \infty$ easily implies (3).

The proof of sufficiency is much more complicated, the paper [17] presents the sufficient stability condition for a particular case of a multi-class model, where inter-retrial times $\xi^{(i)}$, $i = 1,\ldots,K$ have a class-dependent NBU distribution. Note, that an arbitrary random variable $\xi \ge 0$ is called NBU if, for each $x,\, y \ge 0$

$$\mathsf{P}(\xi > x + y|\xi > y) \ge \mathsf{P}(\xi > x). \tag{4}$$

The strict equality in (4) is obtained for the case of exponentially distributed ξ.

Under assumption, that NBU property (4) holds for retrial times $\xi^{(i)}$, $i = 1,\ldots,K$ and $\mathsf{E}\xi^{(i)} < \infty$, denote by F_i the corresponding distribution functions. Then consider an extra distribution function from [17] as follows

$$F_0(x) := \min F_i(x).$$

Let F_0 define a random variable ξ. The relation $F_0(x) \le F_i(x)$ implies the following stochastic inequality [18]:

$$\xi_0 \le_{st} \xi^{(i)}, \quad i = 1,\ldots,K.$$

The sufficient stability condition of such a model is obtained by the following

Theorem 2. [17] *Assume condition (3) holds, the multi-class retrial model under consideration has zero initial state and the inter-arrival time τ is unbounded, that is, for each $x \ge 0$,*

$$\mathsf{P}(\tau > x) > 0. \tag{5}$$

Moreover, assume that each function F_i is NBU distribution such, that

$$\inf(x : F_i(x) > 0) = 0,\ 1 \le i \le K. \tag{6}$$

Then the system is stable, $\mathsf{E}\alpha < \infty$.

The condition (5) automatically implies (2) and always holds for the case of Poisson input, while (6) means, that the retrial time $\xi^{(i)}$ takes arbitrary small value with a positive probability.

In section IV we extend the approach from Theorem 2 to derive the sufficient stability condition of K-class retrial system with general retrials. Thus, we briefly discuss the main steps of the proof of Theorem 2.

The goal is to show that under condition $\rho < 1$, the system is stable. First, we need to show the the negative drift of the orbit size process $N(t)$ as follows. Explore the summary idle time Δ_n of the server oveer the n-th arrival interval. We show that Δ_n goes to zero, as $n \to \infty$ in case the orbit size increases. Thats also an intuitive result, the orbit growth causes more frequent retrials attempt and more aggressive attack to the server. The summary orbit rate goes to infinity, we obtain the continuous secondary stream and the retrial model under consideration starts to behave as infinite buffer queuing system with well-known stability criterion $\rho < 1$.

Thus, the orbit size process is stochastically bounded, which means that $N(t)$ can not increases unlimitedly or visits a bounded set infinitely often. Finally, having in mind (5), we are able to show that starting in the bounded set, the basic process $\{X(t)\}$ (the total number of customers in the system) regenerates, or reaches zero state with a positive probability within a finite interval. Which is equivalent to the result

$$\beta(t_n) \not\Rightarrow \infty,\, n \to \infty, \tag{7}$$

where $\beta(t_n)$ is a residual regeneration time, defined in (1) and "$\Rightarrow$" denotes a weak convergence. By regeneration argument, (7) means positive recurrence of the process X: $\mathsf{E}\alpha < \infty$, and positive recurrence is equivalent to the stability of the system under consideration.

IV. STABILITY ANALYSIS

In this section we present the stability criterion of a multi-class single-server retrial system with Poisson input and general retrials. Namely, we show that such a system is stable if and only if its load coefficient is bounded above by 1. The proof of necessity is based on Theorem 1, while to establish sufficiency we partly rely on Theorem 2 and use some results of Renewal theory. In particulary, we apply Lorden's inequality (see [12]) to establish the negative drift oh the orbit size process. To use the renewal arguments, we additionally assume the fulfillness of moment properties for inter-retrial times as follows:

$$\mathsf{E}\left(\xi^{(i)}\right)^2 < \infty, \qquad i = 1,\ldots,K.$$

Thus, we are able to present the stability criterion for a model under consideration.

Theorem 3. *K-class single-server retrial system with Poisson input and general retrials is stable if, and only if*

$$\rho < 1. \tag{8}$$

Proof:
1. Necessary stability condition. Assume, that system under consideration is positive recurrent. As the input Poisson, τ is unbounded, which yields, that for any arbitrary distributions of service times $\max_i \mathsf{P}(\tau > S^{(i)}) > 0$. Thus, by Theorem 1, $\rho < 1$.
2. Sufficient stability condition. Assume that condition (8) holds. Our goal is to show the stability.

Remind, Δ_n is a summary idle time of the server over τ_n. First, we rely on the basic result from [16], which holds independently on distribution of $\xi^{(i)}$, $i = 1,\ldots,K$. Namely, under condition $\rho < 1$:

$$\mathsf{E}\Delta_n \not\to 0, \quad n \to \infty. \tag{9}$$

Next our goal is to show the summary orbit is stochastically bounded. Assume the opposite case:

$$N_n \Rightarrow \infty, \quad n \to \infty. \tag{10}$$

Then the aim is to show, that assumption (10) implies $\mathsf{E}\Delta \to 0$ and leads to the contradiction with (9).

Denote by D_n the total number of departures over τ_n. Then for some arbitrary constants $d,\, d_0 > 0$ we construct the events

$$A := \{N_n \le d + d_0\}, \quad B := \{D_n \le d_0\}$$

and present the mean idle period as

$$\mathsf{E}\Delta_n = \mathsf{E}[\Delta_n, A] + \mathsf{E}[\Delta_n, \overline{A} \cap \overline{B}] + \mathsf{E}[\Delta_n, \overline{A} \cap B]. \tag{11}$$

Now, the goal is to to establish upper bound of $\mathsf{E}\Delta_n$. First, we construct an auxiliary renewal processes $M_0(t)$, based on intervals distributed as $\min_i S^{(i)}$. Then for K independent replications of $M_0(t)$ (denoted by $M_0^{(i)}(t)$, $i = 1, \dots, K$) we consider

$$M(t) := \sum_{i=1}^{K} M_0^{(i)}(t).$$

The paper [16] gives the following inequalities :

$$\begin{aligned} \mathsf{E}[\Delta_n,\, A] &\le \mathsf{E}\tau \mathsf{P}(N_n \le d + d_0), \\ \mathsf{E}[\Delta_n,\, \overline{A} \cap \overline{B}] &\le a\mathsf{P}(M(a) > d_0) + \mathsf{E}[\tau; \tau > a], \end{aligned}$$

where $a,\, d,\, d_0 > 0$ are arbitrary constants. Thus, we had to bound the last summand from (11):

$$\mathsf{E}[\Delta_n, N_n > d + d_0,\, D_n \le d_0].$$

Note, that condition $\{N_n > d+d_0,\, D_n \le d_0\}$ means, that just before the instant t_n the summary orbit contains more, than $d + d_0$ customers, while not more, that d_0 will be departed up to the moment t_{n+1}. Thus, in this case $N_{n+1} > d$.

Let, for some fixed $d > 1$ and $n \ge 1$: the summary orbit size $N_n = d$ (that holds by assumption (10)). Then define by $\zeta_n(d)$ the time interval since arrival instant t_n until the 1-st, after t_n, retrial attempt. Next, we order the secondary customers by the actual arrival instants and denote such instants by A_j, $j = 1, \dots, d$. For

$$d' := \lfloor d/2 \rfloor \tag{12}$$

define $\eta_n(d')$ – the time since t_n until the 1-st retrial attempt form one of d' the oldest orbit customers (we ignore attempts from other $(d - d')$ orbit customers). Then, for $t \ge A_{d'}$ we construct *a renewal processes*

$$\Lambda_j(t), \quad j = 1, \dots, d',$$

associated with the number of unsuccessful attempts of the j-th secondary. Note, that inter-renewal times of a process $\Lambda_j(t)$ are stochastically equivalent $\xi^{(i_j)}$, under consideration, that the j-th secondary customer belongs to the class $i_j \in \{1, \dots, K\}$. Next, denote by t_n^* – the first *departure* instant after t_n. Note, that if

$$t_n^* > t_{n+1}, \quad \Delta_n = 0 \text{ with probability 1.}$$

Construct $\mathbf{B}_j(t_n^*)$ – residual time from t_n^* to the next renewal in a process $\Lambda_j(t)$. Hence

$$\begin{aligned} \mathsf{P}(\zeta_n(d) > x) &\le \mathsf{P}(\eta_n(d') > x) \\ &= \Pi_{i=1}^{d'} \mathsf{P}(\mathbf{B}_j(t_n^*) > x). \end{aligned}$$

Then we build an extra renewal process $\Lambda_0(t)$ with intervals $\xi^{(0)}$ such, that its residual at instant t_n^* time before the next renewal is constructed according to the following condition:

$$\mathsf{P}(\mathbf{B}_0(t_n^*) > x) = \max_j \mathsf{P}(\mathbf{B}_j(t_n^*) > x), \tag{13}$$

where $\xi^{(0)}$ is distributed as inter-retrial time of corresponding to (13) class $i = 1, \dots, K$. Thus,

$$\mathsf{P}(\zeta_n(d) > x) \le \Big(\mathsf{P}(\beta_0(t_n^*) > x)\Big)^{d'} \le \mathsf{P}(\beta_0(t_n^*) > x).$$

Which leads to

$$\begin{aligned} \mathsf{E}\zeta_n(d) &= \int_0^\infty \mathsf{P}(\zeta_n(d) > x) dx \\ &\le \int_0^\infty \Big(\mathbf{B}_0(t_n^*) > x)\Big)^{d'} dx. \end{aligned} \tag{14}$$

Obviously that for each $x > 0$

$$\Big(\mathsf{P}(\mathbf{B}_0(t_n^*) > x)\Big)^{d'} \to 0,\ d \to \infty.$$

(Recall, that d and d' are related by (12).) Note, for $d' > 1$

$$\Big(\mathsf{P}(\mathbf{B}_0(t_n^*) > x)\Big)^{d'} \le \mathsf{P}(\mathbf{B}_0(t_n^*) > x).$$

Next we apply the key result from renewal theory, by **Lorden's inequality** (see [12]) for $\forall\, t_n^*$:

$$\int_0^\infty \mathsf{P}(\mathbf{B}_0(t_n^*) > x) dx = \mathsf{E}\mathbf{B}_0(t_n^*) \le \frac{\mathsf{E}\Big(\xi^{(0)}\Big)^2}{\mathsf{E}\xi^{(0)}}.$$

As $\mathsf{E}\Big(\xi^{(j)}\Big)^2 < \infty$ for $\forall\, j = 1, \dots, K$, then $\mathsf{E}\Big(\xi^{(0)}\Big)^2 < \infty$ and $\mathsf{P}(\mathbf{B}_0(t_n^*) > x)$ is integrable with respect to x.

Note,

$$\mathsf{E}[\Delta_n, N_n > d + d_0, D_n \le d_0] \le \mathsf{E}\zeta_n(d).$$

Then, by (14) and dominance convergence (Lebesgue):

$$\lim_{d \to \infty} \mathsf{E}\zeta_n(d) \le \lim_{d \to \infty} \int_0^\infty \Big(\mathsf{P}(\mathbf{B}_0(t_n^*) > x)\Big)^{d'} dx = 0.$$

In the event $\{N_n > d + d_0, D_n \le d_0\}$ we obtain at most $(1 + d_0)$ idle periods, hence

$$\begin{aligned} \mathsf{E}\Delta_n &\le \mathsf{E}\tau \mathsf{P}(N_n \le d + d_0) + a\mathsf{P}(M(a) > d_0) \\ &+ \mathsf{E}[\tau; \tau > a] + (1 + d_0)\, \mathsf{E}\zeta_n(d). \end{aligned}$$

Next we choose the values of arbitrary constants as follows:

$$\begin{aligned} a &: \mathsf{E}[\tau; \tau > a] \le \varepsilon_0/4,\ \forall \varepsilon_0 > 0; \\ d_0 = d_0(a) &: a\mathsf{P}(M(a) > d_0) \le \varepsilon_0/4; \\ d = d(d_0) &: \text{so large, that } (1 + d_0)\mathsf{E}\zeta_n(d) \le \varepsilon_0/4. \end{aligned}$$

Under assumption $N_n \Rightarrow \infty$, as $n \to \infty$

$$\exists\, n_1 = n_1(\varepsilon_0) \,:\, \mathsf{E}\tau \mathsf{P}(N_n \le d + d_0) \le \varepsilon_0/4, \quad \forall n \ge n_1.$$

Thus, for an arbitrary $\varepsilon_0 > 0$ and after the appropriate selection of constants $a,\, d_0,\, d,\, n_1$:

$$\mathsf{E}\Delta_n < \varepsilon_0, \ \text{ for all } n \ge n_1,$$

which violates (9). Thus, the assumption (10) was incorrect, and the summary orbit N_n is stochastically bounded for the large n. That actually means that there exist a deterministic subsequence of the arrival instances $t_{n_k} \to \infty$ as $n_k \to \infty$, and constants $\delta > 0$, $N^* < \infty$ such that

$$\inf_k \mathsf{P}(N_{n_k} \le N^*) \ge \delta.$$

As $\mathsf{E}S^{(i)} < \infty$, $i = 1, \dots K$ (yields from $\rho < 1$), the input is Poisson, then exactly as in [16], [17], we are able to obtain, that after an instant t_{n_k} with a positive probability the system becomes idle in a finite interval. Thus, for remaining at instant t_{n_k} regeneration time we have

$$\beta(t_{n_k}) \not\Rightarrow \infty, \quad n_k \to \infty.$$

Hence, the basic regenerative process X is positive recurrent: $\mathsf{E}\alpha < \infty$, which is equivalent, that the system under consideration is stable. ■

Remark. Theorem 3 holds for the system with an arbitrary renewal input, if τ is unbounded: for $x \ge 0$

$$\mathsf{P}(\tau > x) > 0$$

and under the condition

$$\max_{1 \le i \le K} \mathsf{P}(\tau > S^{(i)}) > 0.$$

Note, that obtained sufficient stability condition $\rho < 1$ we successfully verified by simulation in previous works [17], [19] for the two-class retrial systems with Pareto or Weibull distributions of retrial times.

V. SIMULATIONS

Stability criterion $\rho < 1$, obtained in [16] for exponential-retrials case, holds for the system with arbitrary renewal input. In this section we present a few simulation results for two class system *with general retrials* and *bounded inter-arrival times* τ to verify, if the condition $\rho < 1$ guarantees stable orbits.

In particular, consider τ is uniformly distributed on the interval $[x_1, x_2]$. Then fix an arbitrary value $x^* \in [x_1, x_2]$. Next assume, if $\tau_n \in [x_1, x^*]$, the n-th customer belongs to class 1, otherwise – to class 2. Thus, we obtain

$$\lambda = \frac{2}{x_1 + x_2}, \quad p_1 = \frac{x^* - x_1}{x_2 - x_1}, \quad p_2 = \frac{x_2 - x^*}{x_2 - x_1}.$$

Obviously, that $\lambda = \lambda_1 + \lambda_2$. Then, consider Pareto distribution of retrial times $\xi^{(i)}$ with a shape parameter fixed and equal to 1 and denote the corresponding scale parameter by $\kappa_i > 2$, $i = 1, 2$. Hence,

$$\gamma_i = \frac{1}{\mathsf{E}\xi^{(i)}} = \frac{\kappa_i - 1}{\kappa_i}.$$

Finally (for simplicity) we assume exponential class-dependent service times with corresponding rates μ_i.

All experiments were based on $n = 1000$ arrivals. We explored the behavior of mean orbit sizes, corresponding to each of two classes:

$$MN1(n), \; MN2(n)$$

among $m = 10$ independent replications.

Fix values $x_1 = 1$, $x_2 = 4$ and $x^* = 2$. In this case

$$\lambda = 0.400, \; \lambda_1 = 1.33(3), \; \lambda_2 = 0.26(6).$$

Thus, the second class customers are twice intensive, than the first class.

Our goal is to explore orbits behavior in two regimes: $\rho < 1$ and $\rho > 1$. We denote the service rate $\mu_i := 1/\mathsf{E}S^{(i)}$, then fix $\mu_1 = 0.5$ and vary values of μ_2 to obtain the appropriate (stable/unstable) mode. Namely, we present the results, corresponding to two series of experiments:

1. $\mu_2 := 0.4$, $\rho_1 = 0.26(6)$, $\rho_2 = 0.66(6)$, $\rho = 0.93(3)$;
2. $\mu_2 := 0.3$, $\rho_1 = 0.26(6)$, $\rho_2 = 0.88(8)$, $\rho = 1.55(5)$.

Fig. 1 illustrates joint results for $\rho = 0.93$ ("stable mode") and $\rho = 1.16$ ("unstable mode"), while the first class retrials are more intensive, than the second class: $\gamma_1 > \gamma_2$.

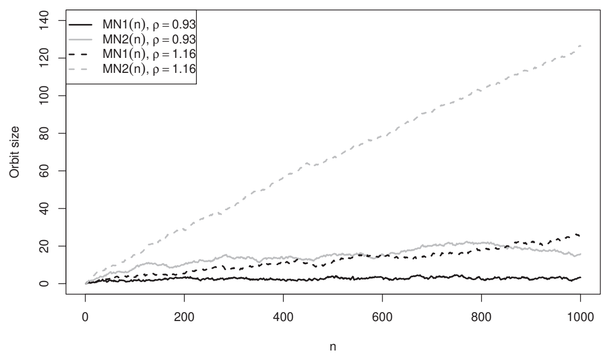

Fig. 1. Orbits dynamics for stable/unstable mode, $\kappa_1 = 1.2$, $\kappa_2 = 3$, $\gamma_1 = 0.67$, $\gamma_2 = 0.17$

For $\rho = 0.93$ both orbits are bounded, while for $\rho = 1.16$ we detected the infinite growth. Thus, we can assume that the the condition $\rho < 1$ guarantees the stability for the single-class retrial system with an arbitrary renewal input. Note, that in both cases the second orbit (grey lines) dominates the first orbit (black lines). Thats an obvious relation, as the first class customers "slower" join the system ($\lambda_1 < \lambda_2$) and then "faster" try to leave the corresponding orbit.

On figure 2 we present the results for symmetric orbit system ($\gamma_1 = \gamma_2$).

The orbits behavior confirms the assumption of stability condition, in both experiments the first orbit is less loaded: $MN1(n) < MN1(n)$ that is explained by input rates relation.

Fig. 3 corresponds to the case of more aggressive the second orbit: $\gamma_2 > \gamma_1$.

In spite the first-class input stream is less intensive, we obtained $MN1(n) > MN2(n)$ for both values of load coefficients. Such a result illustrates the significant influence of retrial rates to whole system behavior. Note that in this configuration the assumption of stability condition holds true.

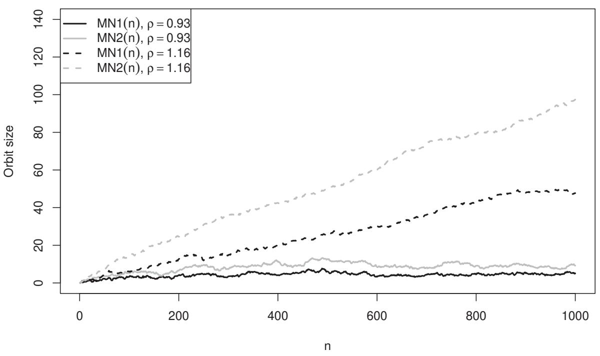

Fig. 2. Orbits dynamics for stable/unstable mode, $\kappa_1 = 1.2$, $\kappa_2 = 1.2$, $\gamma_1 = 0.67$, $\gamma_2 = 0.67$

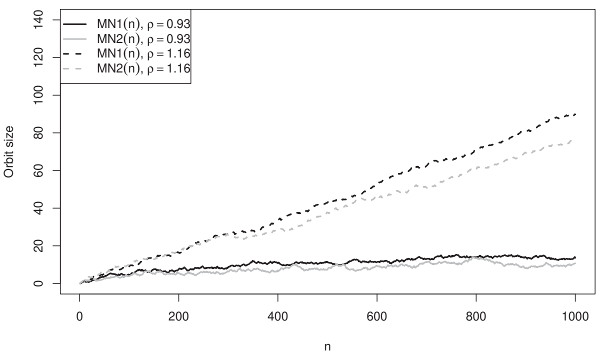

Fig. 3. Orbits dynamics for stable/unstable mode, $\kappa_1 = 3$, $\kappa_2 = 1.2$, $\gamma_1 = 0.17$, $\gamma_2 = 0.67$

VI. Conclusion

The presented research is related to stability analysis of a general single-server retrial system under classical retrial policy. Relying on regenerative approach and previous stability results for some particular models, we obtained that under condition $\rho < 1$, the system with general retrials and Poisson input is stable. The key moment of the proof was based on Lorden's inequality for the embedded renewal process. Simulation results illustrated that obtained stability condition could be extended for a general retrial system with bounded inter-arrival times. The experiments had also shown the great influence of the marginal input rates to the behavior of the whole system. Such a result could be useful in simulation of the system with different priorities of incoming costumers. Setting the appropriate values of orbit rates, we can redistribute the load in whole the system and manage the traffic.

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References

[1] P. Tran-Gia and M. Mandjes, "Modeling of customer retrial phenomenon in cellular mobile networks", *IEEE Journal on Selected Areas in Communications*, vol. 15, Jan.1997, pp. 1406-1414.

[2] S. Aguir, F. Karaesmen, O. Z. Aksin and F. Chauvet, "The impact of retrials on call center performance", *OR Spectrum*, vol. 26, Jul. 2004, pp. 353-376.

[3] G.I. Falin and J.G.D. Templeton, *Retrial Queues. Chapman and Hall*. London, 1997.

[4] J.R. Artalejo and A. Gomez-Corral, *Retrial Queueing Systems: A Computational Approach*. Springer, Cham, 2008.

[5] J. Kim and B. Kim, "A survey of retrial queueing systems", *Annals of Operations Research*, vol. 247, Dec. 2016, pp. 3–36.

[6] T. Phung-Duc, "Retrial Queueing Models: A Survey on Theory and Applications", *ArXiv*, vol. abs/1906.09560, Jun. 2019, pp.1-31.

[7] J.R. Artalejo and T. Phung-Duc, "Single server retrial queues with two way communication", *Applied Mathematical Modelling*, vol. 37, Feb. 2013, pp. 1811-1822.

[8] H. Sakurai, T. Phung-Duc, "Two-way communication retrial queues with types of outgoing calls", *TOP*, vol. 23, Oct. 2015, pp. 466-492.

[9] E. Altman and A.A. Borovkov, "On the stability of retrial queues", *Queueing systems*, vol. 26, Nov. 1997, pp. 343-363.

[10] E. Morozov, "A multiserver retrial queue: Regenerative stability analysis", *Queueing Systems*, vol. 56, May 2007, pp. 157-168.

[11] W. Kang, "Fluid limits of many-server retrial queues with non-persistent customers", *Queueing Systems*, vol. 79 (2), Frb. 2015, pp. 183-219

[12] S. Asmussen, *Applied Probability and Queues*. Wiley: New York, 1987.

[13] E. Morozov, "The tightness in the ergodic analysis of regenerative queueing processes", *Queueing Systems*, vol. 27, Dec. 1997, pp. 179-203.

[14] E. Morozov and R. Delgado, "Stability analysis of regenerative queues", *Automation and Remote control*, vol. 70, May 2009, pp. 1977-1991.

[15] T. Phung-Duc, "Retrial Queueing Models: A Survey on Theory and Applications", *in book Stochastic Operations Research in Business and Industry. Project: Queueing and Performance Evaluation*, May 2017, pp.1-31.

[16] E. Morozov and T. Phung-Duc, "Stability analysis of a multiclass retrial system with classical retrial policy", *Performance Evaluation*, vol. 112, Jun. 2017, pp. 15-26.

[17] E. Morozov, R. Nekrasova R, "Stability Conditions of a Multiclass System with NBU Retrials", *Queueing Theory and Network Applications. QTNA 2019. Lecture Notes in Computer Science*, vol. 11688, 2019, pp. 51-63, doi: 10.1007/978-3-030-27181-7-4.

[18] A. Muller and D. Stoyan, *Comparison Methods for Stochastic Models and Risks. Wiley Series in Probability and Statistics*. John Wiley and Sons Ltd., Chichester, 2002.

[19] R. Nekrasova, "On Verification of Stability of Multi-orbit System with General Retrials: Simulation Approach", *in Communications in Computer and Information Science book series*, vol. 1114, Dec. 2019, pp. 401-412.