Efficient Algorithm of Well-Localized Bases Construction

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Agenda

1. Time-Frequency Localization
2. Localization Dilemma
3. Generalized Weyl-Heisenber bases
4. Construction and optimization ideas
5. Another way and orthogonality conditions
6. Winner basis and finally “fast” algorithm
7. OFDM and OFTDM application

Time-Frequency Localization

The world of transients is considerably larger and more complex
Than the garden of stationary signals.

Stephane Mallat

Fourier transform

- Linear time invariant systems

$$\forall \omega \in \mathbb{R}, Le^{i\omega t} = \hat{\lambda}(\omega)\, e^{i\omega t}$$

$$Lf(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\,\hat{\lambda}(\omega)\, e^{i\omega t} d\omega, \quad F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt$$

Window Fourier transform

- Gabor atoms

$$g_{u,\xi}(t) = g(t-u)\, e^{i\xi t},$$

- $Sf(u,\xi) = \int_{-\infty}^{\infty} f(t)\, g_{u,\xi}^{*}(t)\, dt = \int_{-\infty}^{\infty} f(t)\, g(t-u)\, e^{-i\xi t} dt.$

Localization Dilemma

Heisenberg uncertainty relation

$$\left(\int_{-\infty}^{\infty}(t-\tau)^2\left|g(t)\right|^2 dt\right)\left(\int_{-\infty}^{\infty}(f-\nu)^2\left|G(f)\right|^2 df\right)\geq\frac{\left\|g(t)\right\|^4}{16\pi^2}$$

Balian-Low theorem

- It is impossible to construct orthogonal well-localized basis in Hilbertian spaces with ordinary scalar product without the loss in spectral efficiency

Dilemma Decision

1. Utilization of several initializing functions

2. Hilbertian spaces with real scalar product

$$\left\langle \varphi_k(t-qT), \varphi_i(t-pT) \right\rangle \triangleq \mathrm{Re}\left\{ \int_{-\infty}^{\infty} \varphi_k(t-qT)\varphi_i^*(t-pT)dt \right\} = \delta[k-i]\delta[p-q]$$

Generalized Weyl-Heisenberg basis

$$\mathcal{B} = \{\psi_{k,l}(t) = g(t - l\tau_0)\exp(j\varphi_{l,k})\exp(2\pi jk\nu_0 t)\},$$
$$k = 0,\dots,M-1;\ l = 0,\pm 1,\pm 2,\dots$$

$$\tau_0\nu_0 = 1/2$$

$$s\ n = \sum_{k=0}^{M-1}\left(\sum_{l=0}^{L-1} c_{k,l}^R\,\psi_{k,l}^R\ n + \sum_{l=0}^{L-1} c_{k,l}^I\,\psi_{k,l}^I\ n\right),\ n \in J_N$$

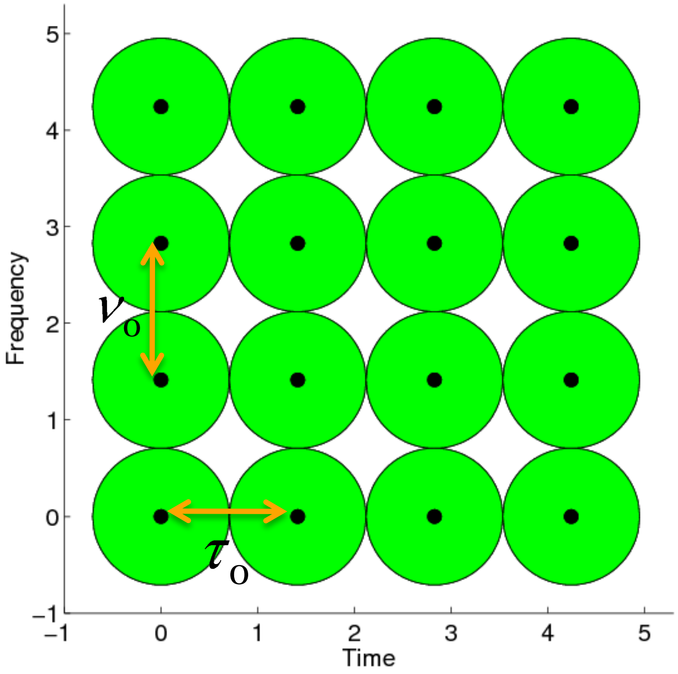
$$\psi_{k,l}^R\ n = g\left[\ n - lM \quad {}_{\bmod N}\right]\exp\left(j\frac{2\pi}{M}k\ n - \alpha/2\ \right),$$

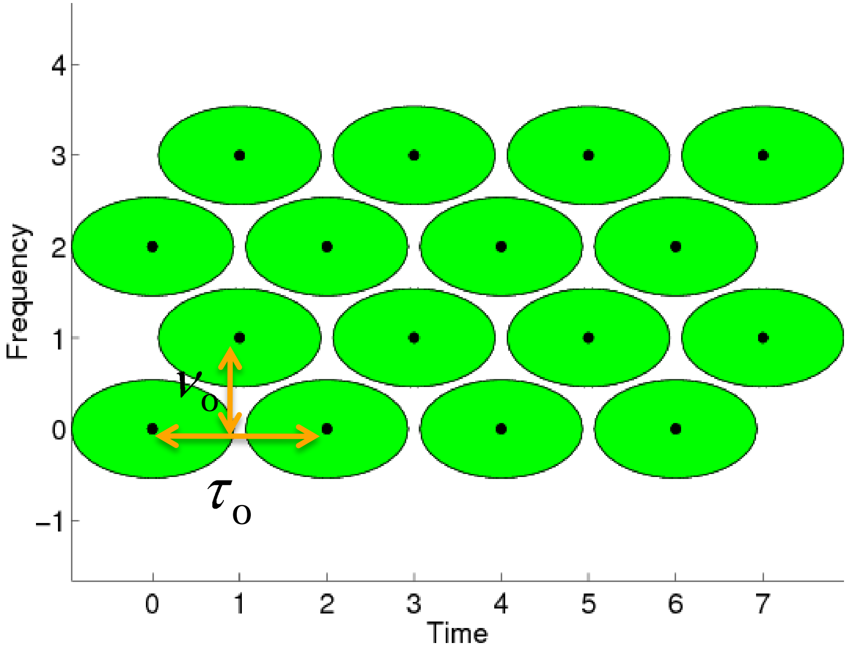
$$\psi_{k,l}^I\ n = -jg\left[\ n + M/2 - lM \quad {}_{\bmod N}\right]\exp\left(j\frac{2\pi}{M}k\ n - \alpha/2\ \right),$$

$$\mathbf{U} = \ \mathbf{U}_R, \mathbf{U}_I$$

$$\mathcal{B}\ J_N \triangleq \ \psi_{k,l}^R\ n\ , \psi_{k,l}^I\ n\quad ,$$

Basic functions packing in time-frequency lattice





Optimal W-G basis (general approach)

Gaussian initializing impulse with necessary parameters

$g_G(t) \longrightarrow g_G(n)$

Symmetrical N-periodical approximation

$g_G(n) \longrightarrow \tilde{g}_G(n)$

WG-basis parameters initialization: M, L, α

Optimal initializing impulse construction (for WG basis)

$g_o(n): \min_{g \in \mathcal{U}} \|\tilde{g}_G - g\|_E^2 \triangleq \varepsilon$ + Orthogonality conditions

Optimal WG-basis parameters search

$\widehat{\alpha}, \widehat{M}, \widehat{L} : \min_{\alpha, M, L} \varepsilon$

Orthogonal matrix of WG basis

$\mathbf{U} = \{g_o^{(n,n)}\}_{n=1}^{N}, N = KL$

Fast signal filtering algorithm

Basis matrix **U** as the multiplication of sparse matrixes

Construction Ideas

Closeness to the Gabor basis (Gaussian initializing functions)

$$\begin{cases} \mathrm{Re}(\mathbf{U}^{*}\mathbf{U}) = \mathbf{I}_{2N} \\ \mathbf{U}_{onm} : \min\limits_{U \in \mathfrak{A}} \|\mathbf{G} - \mathbf{U}\|_{E}^{2}, \end{cases}$$

To derive orthognality conditions directly for the initializing function

$$\sum_{r=0}^{2L-1} g\left[\left(n - r\,{}^{M}\!/_{2}\right)_{N}\right] g\left[\left(n - r\,{}^{M}\!/_{2} - lM\right)_{N}\right] = \frac{2}{M}\delta_{l,0}, \forall n \in \{0, \ldots, N-1\}$$

Winner Transform

- Winner Transform

$$\eta_k(n) = \sum_{r=0}^{2L-1} g\left[\left(n - r\, {}^{N}\!/_{2L}\right)_N\right] \exp\left(\frac{2\pi j}{2L} rk\right).$$

- Inverse Winner transform

$$g(n) = \frac{1}{2L} \sum_{k=0}^{2L-1} \eta_k(n).$$

- WG basis orthogonality condition

$$\left|\eta_k^{M/2}(n)\right|^2 + \left|\eta_{k+L}^{M/2}(n)\right|^2 = {}^{4}\!/_{M}.$$

Fast construction algorithm

$$\eta_k^{M/2}\ n = \frac{2\tilde{\eta}_k^{M/2}\ n}{\sqrt{M\left|\tilde{\eta}_k^{M/2}\ n\right|^2 + M\left|\tilde{\eta}_{k-L}^{M/2}\ n\right|^2}}, \qquad \tilde{\eta}_k^{M/2}\ n = \sum_{r=0}^{2L-1} g_o\left[\ n - rM/2\ _N\right]\exp\left(\frac{2\pi j}{2L}rk\right)$$

Construct matrix **Z** from the vector of shifts of g0 as columns $\vec{g}_0\ n \triangleq\ g_0\ n\ , g_0\left[\ n - N/2L\ _N\right], \ldots, g\left[\ n - \ 2L-1\ N/2L\ _N\right]^T$

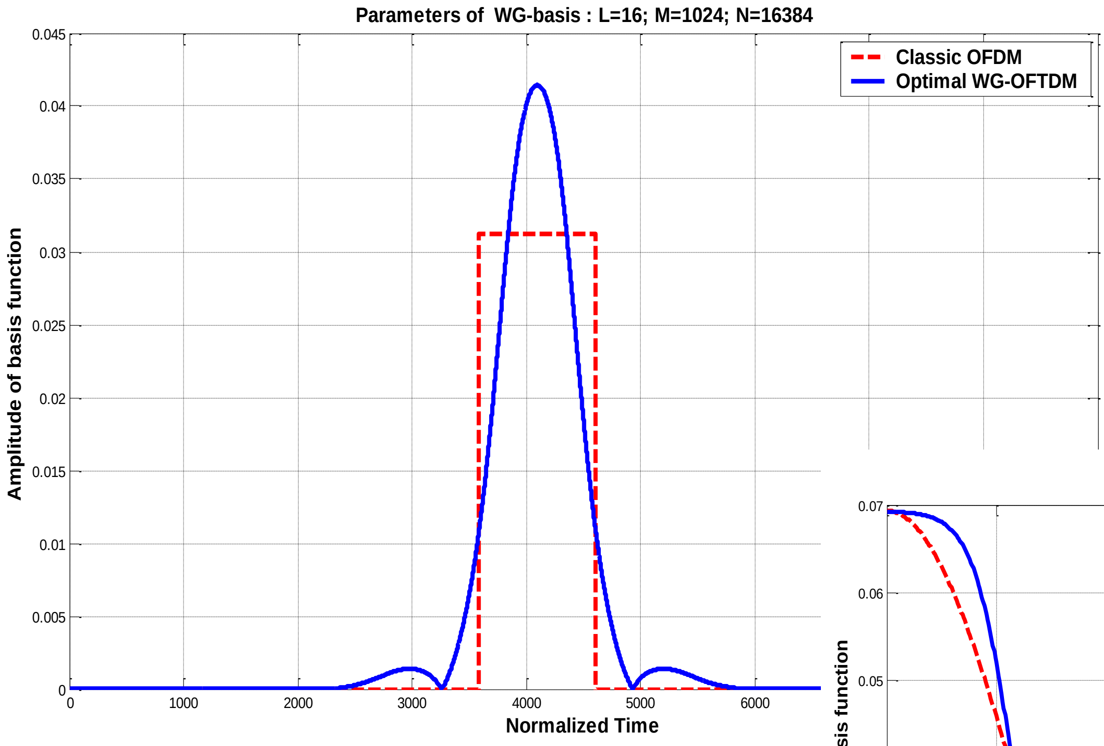
Winner transform on g0

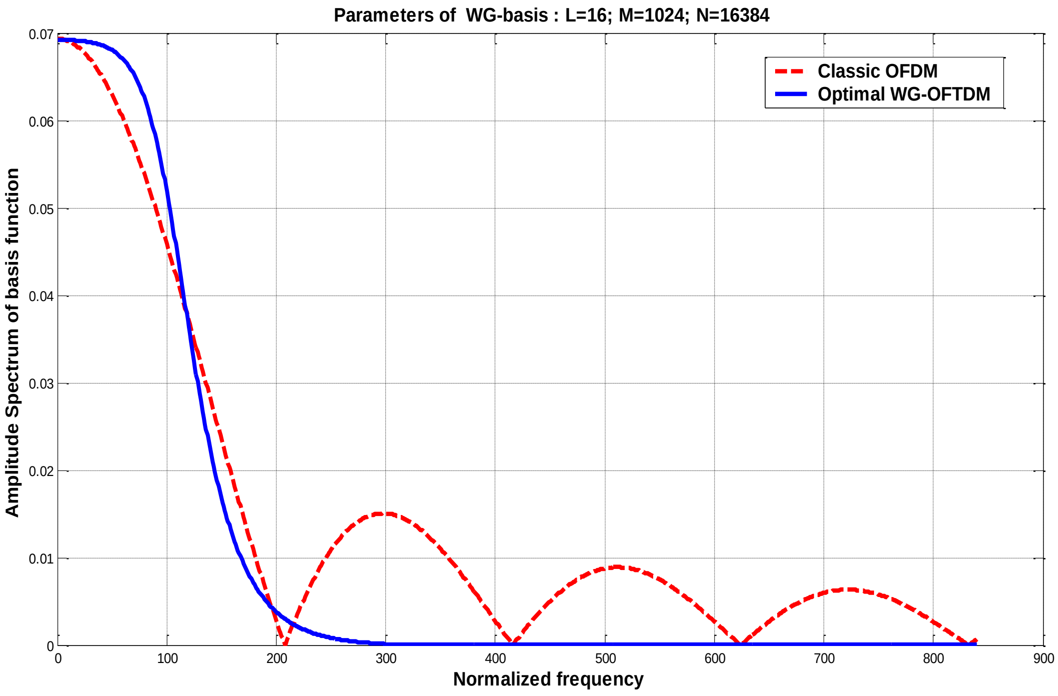
<-> FFT on matrix **Z** rows

Construct Winner basis $\eta_k^{M/2}\ n$

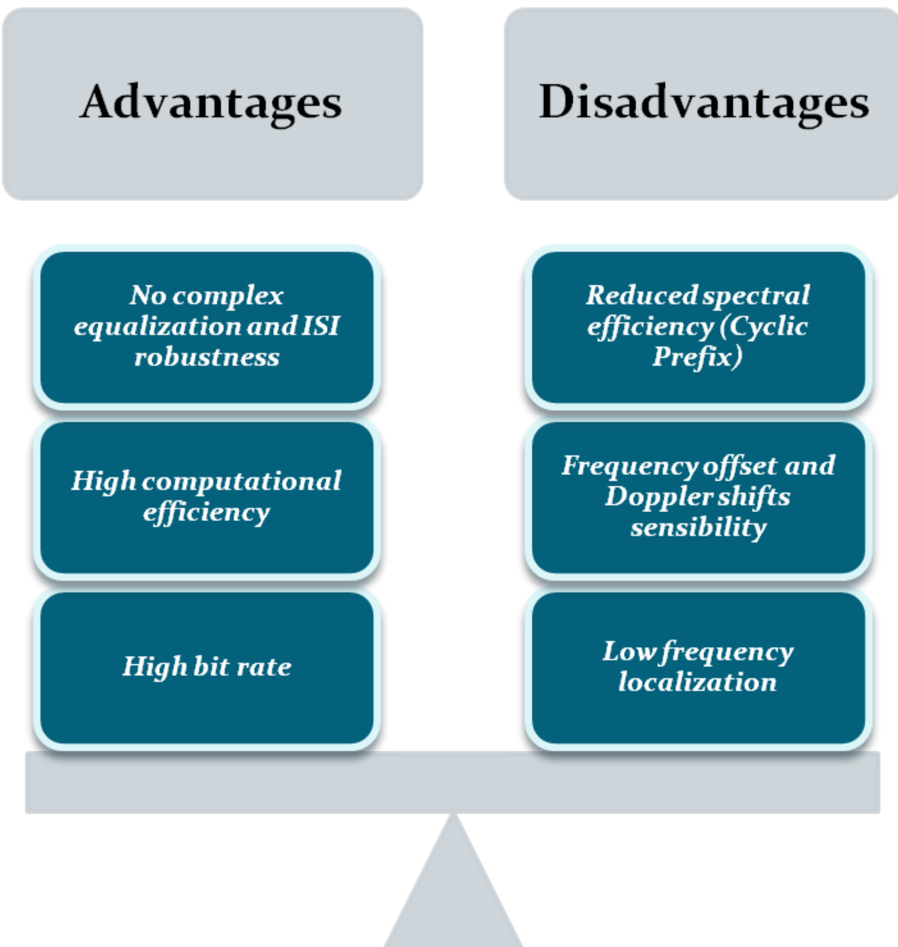
Inverse Winner transform

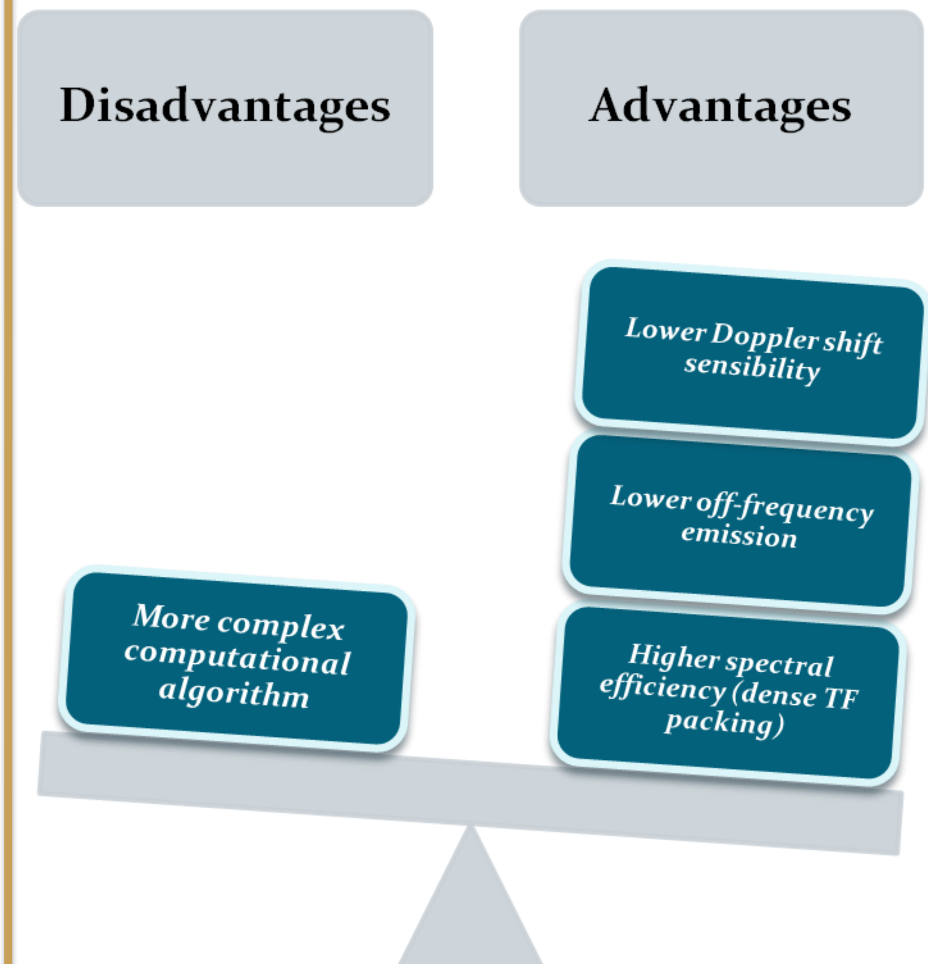
Modeling results





OFDM and OFTDM





Q&A

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